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Lambda Calculus
Winter Term 2023/24
Solutions to Exercise Sheet 8

Exercise 1 (Progress Property)

Let t be a closed and well-typed term, i.e. $[] \vdash t : \tau$ for some τ. Show that t is either a value or there is a t' such that $t \rightarrow_{cbv} t'$.

Solution

The proof follows an induction on the derivation of $[] \vdash t : \tau$.

Case $[] \vdash x : \tau$
This case cannot occur since x does not have a type in the empty environment.

Case $[] \vdash (\lambda x.\ s) : \tau_1 \rightarrow \tau_2$
Then, $(\lambda x.\ s)$ is already a value.

Case $[] \vdash t_1\ t_2 : \tau_2$ **where** $[] \vdash t_1 : \tau_1 \rightarrow \tau_2$ **and** $[] \vdash t_2 : \tau_1$
By the induction hypothesis, both t_1 and t_2 can take a step or are a value. If t_1 can take a step, we can use the left application rule on t. If t_1 is a value and t_2 can take a step, then the right application rule can be used. If t_1 and t_2 are both values, we know $t_1 = \lambda x.\ t_1'$ for some t_1' as t_1 is of type $\tau_1 \rightarrow \tau_2$. Thus we can apply the rule for reducing an abstraction.

Exercise 2 (Normal Form)

Show that every type-correct $\lambda^{\rightarrow}$-term has a β-normal form.

Solution

We prove the statement by coming up with a terminating reduction relation $\rightarrow_p$ that, when repeatedly applied, reduces a given term to β-normal form. Furthermore, we define a well-founded order $<_T$ on terms and show that $t_1 \rightarrow_p t_2$ implies $t_1 <_T t_2$. By induction on $<_T$ it then follows that $\rightarrow_p$ is terminating.

The reduction strategy is chosen such that it decreases the types of subterms.

Let $|\tau|$ be the size of a type τ, i.e. the number of function-arrows occuring in τ.

$$|\alpha| = 0$$
$$|\alpha \rightarrow \beta| = |\alpha| + |\beta| + 1$$

With this measure, we can assign a natural number to each β-redex:

$$|(\lambda x.\ s)\ t|^{\Gamma} = |\tau_1 \rightarrow \tau_2| \quad \text{where} \quad \Gamma \vdash (\lambda x.\ s) : \tau_1 \rightarrow \tau_2$$

We assign each term t a multiset M_t. In order to account for the potentially non-empty environment Γ in the subterms of t, we first define M_t^Γ recursively and then set $M_t := M_t^{[]}$.

$$\begin{aligned} &t = u\ v \Longrightarrow M_t^\Gamma = M_u^\Gamma \cup M_v^\Gamma \cup \{|u\ v|^\Gamma.\ u\ v \text{ is a } \beta\text{-redex}\} \\ &t = (\lambda x.\ s) \Longrightarrow \Gamma \vdash (\lambda x.\ s) \colon \tau_1 \to \tau_2 \Longrightarrow M_t^\Gamma = M_s^{\Gamma[x\colon\tau_1]} \\ &t = x \Longrightarrow M_t = \{\} \end{aligned}$$

We can view multisets as functions into the natural numbers and define an ordering on them:

$$\begin{aligned} M <_M N \iff & M \neq N \wedge \\ & (\forall y.\ M(y) > N(y) \Longrightarrow (\exists x.\ x > y \wedge M(x) < N(x))) \end{aligned}$$

It can be proved that the multiset ordering terminates (is well-founded). The ordering naturally extends to terms, i.e. $u <_T v \iff M_u <_M M_v$.

If one regards a β-redex of the form $r = (\lambda x.\ u)\ v$ with $\Gamma \vdash r : \tau$ and u and v in β-NF, then we have $M_r >_M M_{r'}$ for the reduct $r' = u[v/x]$.

This is because although the substitution may create new β-redexes w, we have $|w| < |r|$ for all those w in r':

> Note that $\Gamma \vdash (\lambda x.\ u) : \tau_1 \to \tau_2$ and $\Gamma \vdash v : \tau_2$ for some τ_1, τ_2 must hold.
> Since $v \in$ NF, w is of the form $(v\ v')$ with $v = (\lambda x.\ s)$ for some s
> and thus $|w| = |\tau_1| < |\tau_1 \to \tau_2| = |r|$.

Thus, if we choose a reduction strategy $\to_p$ that reduces an innermost β-redex in t, we have:

$$t \to_p t' \Longrightarrow M_t >_M M_{t'}$$

We can obtain such a reduction strategy by restricting the first rule of $\to_\beta$ to:

$$\frac{s \in \text{NF} \qquad t \in \text{NF}}{(\lambda x.\ s)\, t \to_p s[t/x]}$$

As the multiset ordering terminates, also the chosen reduction strategy must terminate. If it terminates with t', then t' is in β-NF because otherwise t' would contain a regex and therefore an innermost regex that can be reduces with $\to_p$.

$\square$

Homework 3 (Typing)

a) Prove:

$$[] \vdash (\lambda x\colon \tau_2 \to \tau_3.\ \lambda y\colon \tau_1 \to \tau_2.\ \lambda z\colon \tau_1.\ x\,(y\,z))\colon (\tau_2 \to \tau_3) \to (\tau_1 \to \tau_2) \to \tau_1 \to \tau_3$$

b) Give suitable solutions for $?\tau_1$, $?\tau_2$, $?\tau_3$ and $?\tau_4$ and prove that the term is type-correct given your solution.

$$[] \vdash \lambda x\colon ?\tau_1.\ \lambda y\colon ?\tau_2.\ \lambda z\colon ?\tau_3.\ \ x\ y\ (y\ z)\ \colon ?\tau_4$$

Homework 4 (β-reduction preserves types)

A type system has the *subject reduction property* if evaluating an expression preserves its type. Prove that the simply typed λ-calculus ($\lambda^{\to}$) has the subject reduction property:

$$\Gamma \vdash t\colon \tau \wedge t \to_\beta t' \Longrightarrow \Gamma \vdash t'\colon \tau$$

Hints: Use induction over the inductive definition of $\to_\beta$ (Def. 1.2.2). State your inductive hypotheses precisely – it may help to introduce a binary predicate $P(t, t')$ to express the property you are proving by induction. Also note that the proof will require *rule inversion*: Given $\Gamma \vdash t\colon \tau$, the shape of t (variable, application, or λ-abstraction) may determine which typing rule must have been used to derive the typing judgment.

Within your proof, you are free to use the following lemma about substitution:

$$\Gamma \vdash u\colon \tau_0 \wedge \Gamma[x\colon \tau_0] \vdash t\colon \tau \Longrightarrow \Gamma \vdash t[u/x]\colon \tau \tag{1}$$

Homework 5 (Implementation of multiset-ordering and reduction)

Implement the multiset ordering and the reduction strategy from the second tutorial exercise in your favorite programming language.