Logics Exercise

TU München
Institut für Informatik

Prof. Tobias Nipkow
Dr. Peter Lammich
Simon Wimmer

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Submission of Homework: Before tutorial on June 1

Exercise 7.1. [QE for DLO]
Use the quantifier-elimination procedurefor DLOs to check wether the following formula is a member of $Th(DLO)$:

$$\exists x \forall y \exists z((x < y \lor z < x) \land y < z)$$

Exercise 7.2. [Compactness for FOL]
Prove the Compactness Theorem for first-order logic:

If every finite subset of a (countable) set M of formulas has a model, then M as a whole has a model.

Hint: You may use the following definitions and theorems:

Let $M = \{F_1, F_2, \ldots\}$. Then $skolem(M) = \{F'_1, F'_2, \ldots\}$, where F'_i is a Skolem form of F_i, and the Skolem-functions in the F'_i are pairwise disjoint.

Note: Wlog, we assume that M is of a form that allows us to find enough fresh function symbols for Skolem functions.

Theorem: M is satisfiable iff $skolem(M)$ is satisfiable.

Exercise 7.3. [Axiomatizations and Compactness]
Using compactness, show that if a theory is finitely axiomatizable, any countable axiomatization of it has a finite subset that axiomatizes the same theory. In other words, if $Cn(\Gamma) = Cn(\Delta)$ with Γ countable and Δ finite, then there is a finite $\Gamma' \subseteq \Gamma$ with $Cn(\Gamma') = Cn(\Gamma)$.

Homework 7.1. [Theories] (5 points)

1. Show $Cn(S) = Th(Mod(S))$, i.e. show $Th(Mod(S)) = \{F \mid F\ \Sigma\text{−sentence and } S \models F\}$
2. Show that Cn is a closure operator, i.e. Cn fulfills the following properties:
   - $S \subseteq Cn(S)$
   - if $S \subseteq S'$ then $Cn(S) \subseteq Cn(S')$
   - $Cn(Cn(S)) = Cn(S)$

Homework 7.2. [Quantifier Elimination for $Th(\mathbb{N}, 0, S, =)$] (5 points)
Give a quantifier-elimination procedure for $Th(\mathbb{N}, 0, S, =)$ where S is the successor operation on natural numbers, i.e. $S(n) = n + 1$.

Hint: $a = b$ iff $S^k(a) = S^k(b)$ for any $a, b, k \in \mathbb{N}$.

Homework 7.3. [Quantifier Elimination for DLOs with endpoints] (5 points)
Let $\Sigma = \{a, b, <, =\}$ and replace the last two axioms for DLOs with:

- $\forall x(x = a \lor a < x)$
- $\forall x(x = b \lor x < b)$

Modify the quantifier-elimination procedure for dense linear orders to obtain a quantifier-elimination procedure for this theory.

What happens if there is only one endpoint?

Homework 7.4. [Decidable Axiomatizations] (5 points)
Show that any set of sentences that is axiomatized by a recursively enumerable set is also axiomatized by a decidable set.

Hint: For each $n \in \mathbb{N}$ a possible encoding of n as a formula could be of the form

$$\underbrace{F \land \ldots \land F}_{n \text{ times } F}$$

for some formula F.