An Inductive Proof of the Wellfoundedness of the Multiset Order

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The following note presents an inductive proof of the wellfoundedness of the multiset order due to Wilfried Buchholz[1] communicated to me[2] by Ralph Matthes[3]. All typos are entirely mine.

1 Wellfounded part

Given a binary relation $<$ on a set S, the subset W of S called the **wellfounded part** of S w.r.t. $<$ is defined inductively as follows [1]:

$$\frac{\forall y < x.\ y \in W}{x \in W}$$

The corresponding induction principle easily yields the principle of **wellfounded part induction**:

$$\frac{\forall x \in W.\ (\forall y < x.\ P(y)) \Rightarrow P(x)}{\forall x \in W.\ P(x)}$$

It also follows that $<$ is wellfounded iff $W = S$.

2 The proof

Let $<$ be a wellfounded relation on a set A, and let $\mathcal{M}(A)$ be the set of all finite multisets over A. We use set-notation for multisets. The letters a and b range over A; K, M and N range over $\mathcal{M}(A)$. We define the following

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abbreviations:

$$
\begin{array}{rcl}
M < a & \Leftrightarrow & \forall b \in M.\ b < a \\
N <_{mult} M & \Leftrightarrow & \exists M_0, a, K.\ M = M_0 \cup \{a\} \wedge N = M_0 \cup K \wedge K < a \\
W & = & \text{the wellfounded part of } \mathcal{M}(A) \text{ w.r.t. } <_{mult}
\end{array}
$$

Lemma 2.1 *If* $\forall b < a.\ \forall M \in W.\ M \cup \{b\} \in W$ *and* $M_0 \in W$ *and* $\forall M <_{mult} M_0.\ M \cup \{a\} \in W$ *then* $M_0 \cup \{a\} \in W$.

Proof by definition of W. Let $N <_{mult} M_0 \cup \{a\}$. We need to prove $N \in W$. There are two possibilities why $N <_{mult} M_0 \cup \{a\}$ holds:

- If $N = M \cup \{a\}$ for some $M <_{mult} M_0$ then $N \in W$ follows from the third assumption.
- If $N = M_0 \cup K$ for some $K < a$ then $N \in W$ follows from the first two assumptions by induction on the size of K.

□

Lemma 2.2 *If* $\forall b < a.\ \forall M \in W.\ M \cup \{b\} \in W$ *then* $\forall M \in W.\ M \cup \{a\} \in W$.

Proof From Lemma 2.1 by wellfounded part induction. □

Lemma 2.3 $\forall M \in W.\ M \cup \{a\} \in W$.

Proof From Lemma 2.2 by wellfounded induction on a. □

Theorem 2.4 $M \in W$.

Proof by induction on the size of M. The base case $\emptyset \in W$ holds because there is no $N <_{mult} \emptyset$. Lemma 2.3 covers the induction step. □

Thus we know that $<_{mult}$ is wellfounded on all of $\mathcal{M}(A)$.

3 Termination

As is well known, wellfoundedness is classically equivalent with termination, i.e. the absence of infinite descending chains. Taking the classical perspective, we can turn things around and define W as the set of all terminating elements of S. Now wellfounded part induction is simply good old wellfounded induction, and the inductive characterization of W is now a consequence of this direct definition of W.

References

[1] P. Aczel. An introduction to inductive definitions. In J. Barwise, editor, *Handbook of Mathematical Logic*. North-Holland, 1977.